Transmission (of anything) in a Finite Population

Why the number of subscribers to CDS does not grow exponentially

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Abstract

This is a very brief note on a simple dynamical model for the transmission of something, here considered to be a piece of information, among a finite population of individuals. As the subtitle hints, this was prompted by a suggestion, many years ago, that the number of users of data and software tools offered by our Chemical Database Service (CDS) should be exponential. The user figures at the time, though increasing nicely, were not exponential; perhaps the suggestion was that we should be working harder to disseminate the word about CDS more effectively. No! The general expectation of exponential growth can only be partially and temporarily fulfilled.

The Covid 19 pandemic and lockdown, with its pretty obviously analogies in transmission dynamics, brought this back to mind. So here is the very simple solution to the very simple model considered. The afterword on the discrete variant – the so-called logistic map – is a curiosity from the point of view of this note, but quite fun.

1 The problem

Consider the transmission of a piece of information (the info) among a population of individuals. In the CDS example, this info is the knowledge that a useful collection of chemical data can be accessed. In biological examples, the info could be a gene, a virus (the RNA in a virus), something like that. In social examples, the info could be an internet meme, whatever that is. It's obviously a simple idea with many and various applications.

2 The model

Let's make the following assumptions.

(a) The total population is *N*, consisting of 2 types: A and B. The population of A/B at time *t* is $n_{A/B}(t)$, where

$$n_A(t)+n_B(t)=N \tag{2.1}$$

(b) The A's know a piece of info; the B's don't. If an A meets an A, or if a B meets a B, no transmission of info occurs. If a B meets an A, the info will be transmitted with probability *p* ($0\le p\le 1$).

(c) In a time interval $\tau\ge 0$, the number of individuals of either type encountered by any given individual is *Q*. Suppose these encounters are random – independent of type. Obviously *Q* depends on the interval τ; assume the dependence is linear with a constant of proportionality $q\ge 0$:

$$Q(\tau)=q\tau \tag{2.2}$$

The populations of the 2 types are related by (2.1). At time *t*, the concentration of A is , so that

$$n_A(t)=c(t)N, \quad n_B(t)=(1-c(t))N \tag{2.3}$$

Clearly $0\le c(t)\le 1$. The problem, therefore, is to determine the single function $c(t)$.

3 The solution

In the interval $t_i\le t\le t_{i+1}=t_i+\tau$, the number of times a B individual encounters another individual is *Q*. There are $n_B(t)$ individuals, so the total number of times a B encounters another individual is $Qn_B(t)$. Assumption (c), randomness, means that the total number of times a B meets an A in the interval is $Qn_B(t)c(t)$. Thus,

$$n_B(t_i+\tau)=n_B(t_i)+pQn_B(t_i)c(t_i)$$

Using (2.3), we have

$$(1-c(t_i+\tau))=(1-c(t_i))+pQc(t_i)(1-c(t_i)) \tag{3.1}$$

Or, putting $t_i=t$,

$$c(t+\tau)-c(t)=pQc(t)(1-c(t)) \tag{3.2}$$

To get a differential equation for $c(t)$, we let τ become small and write

$$c(t+\tau)-c(t)=\tau\frac{dc(t)}{dt}+O(\tau^2)$$

From (2.2), we obtain

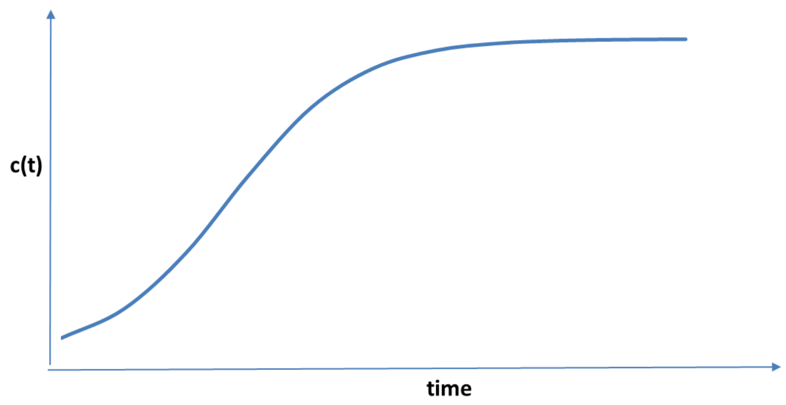
$$\tau\frac{dc(t)}{dt}=pq\tau c(t)(1-c(t))+O(\tau^2)$$

Thus, as $\tau\to 0$

$$\dot{c}(t)\equiv\frac{dc(t)}{dt}=rc(t)(1-c(t)) \\ r\equiv pq\geq 0 \tag{3.3}$$

We recognise this as a version of the famous logistic equation of population dynamics, at least if we have recently read, for example, Nowak's book[1] [1].

Note that when $c(t)$ is small, we have $\dot{c}(t)\sim rc(t)$, which implies that $c(t)\propto e^{rt}$: the initial growth of the informed population is indeed exponential. Similarly, as $c(t)$ approaches 1, we find that $\dot{c}(t)\sim r(1-c(t))$, which implies that $c(t)\propto 1-e^{-rt}$. Therefore, the graph of $c(t)$ must look something like this

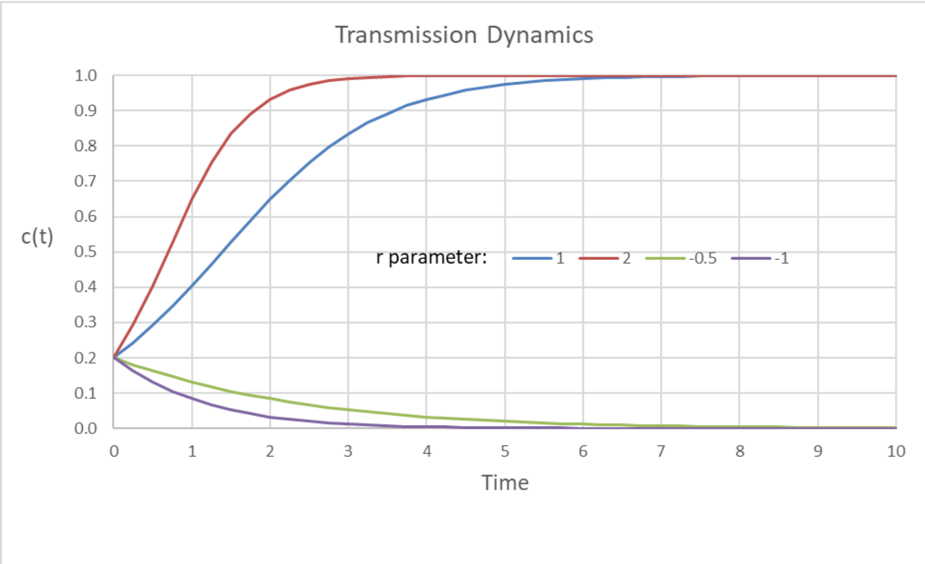


The exact solution of (3.3), as can easily be verified by direct differentiation, is

[1] This is a nice introduction, particularly interesting reading during the Coronavirus lockdowns, although Nowak's (non-scientific) reputation has been damaged recently (see Wikipedia).

$$c(t)=\frac{c_0 e^{rt}}{1+c_0\left(e^{rt}-1\right)} \quad (3.4)$$

Here, $c_0 = c(t=0))$ is the initial value of the concentration of A individuals; obviously, if c_0 =0 no transmission of info can occur, because there are no informed individuals – ignorance continues to reign.



This graph shows solutions for two negative values (-0.5 and -1) of the rate of transmission parameter r, in spite of the remark in (3.3) that r must be positive. I'm thinking here of the possibility of the loss of info – individuals might forget what they've been told[2] at a rate $l>0$. The effect of this would be to modify the r parameter in (3.3) in the obvious way:

$$\begin{aligned} & r \to r' = r - l = l(r/l - 1) = l(R-1) \\ & \dot{c}(t) = r'c(t)\left(1-c(t)\right) \end{aligned} \quad (3.5)$$

So, consider the parameters in the legend of the above graph to represent r', which can obviously be negative if the rate of loss is greater than the rate of transmission. Equation (3.5) indicates that the initial behaviour is either exponential growth or exponential decay, depending on whether $R = r/l$ is greater or less than 1. I suppose that this is essentially what is meant by the famous "R number" so terribly familiar to us all now, over a year into the Covid 19 pandemic.

4 Afterword – the logistic map

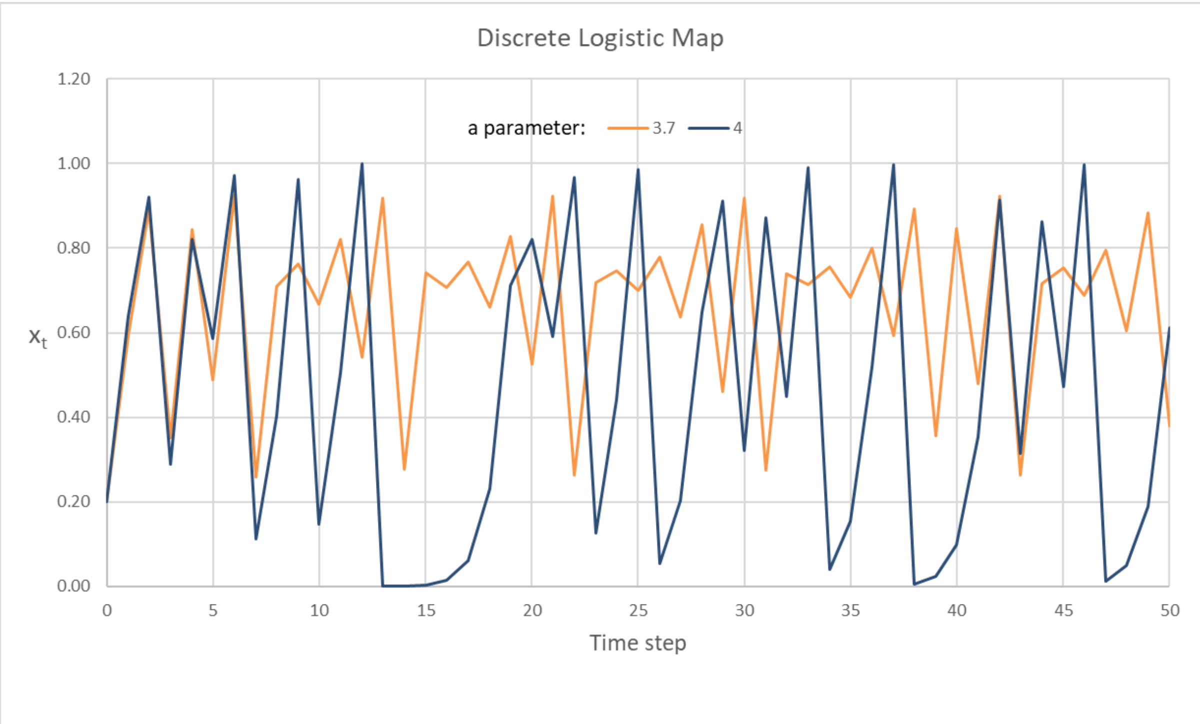
Discretise time into steps to obtain a difference equation

$$x_{t+1} = ax_t\left(1+x_t\right) \quad (4.1)$$

[2] The analogue of forgetting in the epidemiology context would be death of some infective agents (bacteria, virus particles etc).

Surprisingly, this leads to the topic of deterministic chaos and Robert May's famous comment [2] that models obeying simple rules may have very complicated dynamics.

Obviously, the map (4.1) is extremely simple – simple enough to implement in a spreadsheet. Here are some results.



These graphs exemplify some of the following properties of the logistic map.

For $0 < a < 1$: the only stable equilibrium solution (ie the value to which x_t tends as the time increases, whatever the initial value of x) is $x = 0$.

For $1 < a < 3$: the stable equilibrium value is $x = (a-1)/a$.

For $3 < a < 4$: the equilibrium is unstable. When a is slightly above 3, x oscillates with period 2. As a increases further, the period of oscillation doubles to 4, then doubles again to 8, and so on. Beyond $a \sim 3.57$, there is an infinite number of even period oscillations and odd periods set in. In brief, all hell breaks loose.

For $a < 0$ and for $a > 4$, unphysical (or unbiological) negative solutions exist.

Wikipedia has an interesting write-up on both the logistic function (essentially the right hand side of (3.4)) and the logistic map. The above figures are just two examples of the very complicated full set of solutions for the logistic map. But that is another story.

References

[1] "Evolutionary Dynamics", Martin A Nowak (Belknap/Harvard, 2006)

[2] "Simple mathematical models with very complicated dynamics", Robert May, *Nature* **261**(5560) 459 (1976)